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Poincaré-Cartan Integral Variants and Invariants of Nonholonomic Constrained Systems

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Usually there does not exist an integral invariant of Poincaré-Cartan's type for a nonholonomic system because a constraint submanifold does not admit symplectic structure in general. An integral variant of Poincaré-Cartan's type, depending on the nonholonomy of the constraints and nonconservative forces acting on the system, is derived from D'Alembert-Lagrange principle. For some nonholonomic constrained mechanical systems, there exists an alternative Lagrangian which determines the symplectic structure of a constraint submanifold. The integral invariants can then be constructed for such systems.

1. INTRODUCTION

Poincaré and Poincaré-Cartan integral invariants of dynamical systems have found important applications in quantum theory, analytical mechanics, statistical mechanics, and hydrodynamics. Usually the integral invariants exist for conservative holonomic systems because such systems admit a natural *symplectic* structure (Arnold, 1978; Liu, 1991; Marsden, 1994). The extension of the theory of integral invariants to *nonholonomic* constrained systems was made by Li and Li (1990). However, they introduced integral invariants of Poincaré-Cartan's type for nonholonomic constrained systems by improperly extending the Hamilton's stationary principle to nonholonomic constrained systems. As is well known, the equations of motion for the nonholonomic constrained systems, such as Routh's equations and Chaplygin's equations, obey D'Alembert-Lagrange's principle instead

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of Hamilton's stationary principle. These two kinds of principles are not equivalent for nonholonomic constrained systems (Cardin and Favretti, 1996; Marle, 1998; Mei, 2000).

In this paper, we prove that in the original choice of dynamical function and coordinates with physical meaning, there exists *an integral variant* of Poincaré-Cartan's type, depending on the nonholonomy of the constraints and nonconservative forces acted on the nonholonomic systems. In this sense, the integral invariants for constrained systems exist if and only if the constraints are integral and the forces acted on the systems are conservative. In a previous paper (Guo *et al.*, 1999), we have pointed that the existence of Poincaré-Cartan integral invariants of a dynamical system is closely related with Lagrangian inverse problem. If a Chaplygin's nonholonomic constrained system whose equations of motion decouple with the constraints satisfies the Helmholtz's conditions (Cariñena and Rañada, 1999; Morando and Vignolo, 1998), there exists an alternative Lagrangian for such a system. An alternative Lagrangian can also be constructed if a nonholonomic constrained system is of a kind of adjoint symmetry (Sarlet *et al.*, 1995). For these cases, the constraint submanifolds are of symplectic structure defined by the alternative Lagrangians. Therefore, Poincaré and Poincaré-Cartan integral invariants can be constructed for such nonholonomic constrained systems.

In Section 2, an integral variant of Poincare-Cartan's type for a nonholonomic nonconservative system is derived from D'Alembert-Lagrange's principle. In Section 3, we introduce integral invariants based on Lagrangian inverse problem for nonholonomic systems. An example is illustrated in the last section.

2. INTEGRAL VARIANTS OF NONCONSERVATIVE NONHOLONOMIC SYSTEMS

We denote the underlying mathematical model of nonconservative or time-dependent mechanics by a contact manifold $R \times TM$ where M is a n-dimensional configuration manifold with local coordinates $\{t, q^i\}$ $(i = 1, 2, \ldots, n)$. Suppose that the system be subjected to affine nonholonomic constraints

$$f^\alpha = A_i^\alpha(t, q^j)\dot{q}^i + A^\alpha(t, q^i) = 0, \quad (\alpha = 1, 2, \ldots, g, g < n) \tag{1}$$

where the matrix (A_i^α) is of maximum rank. The conservative part of the system is described by a continuous, regular Lagrangian $L : R \times TM \to R$, that is, $L \in \mathfrak{L}^2(TM \times R)$, $\partial^2 L/\partial\dot{q}^i\partial\dot{q}^j \neq 0$. The components of nonconservative generalized forces are represented by $Q_i(t, q^j, \dot{q}^j)$.

We begin with the D'Alembert-Lagrange principle

$$\left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + Q_i\right)\delta q^i = 0 \tag{2}$$

where δq^i are virtual displacements and the Einstein's sumation convention is understood. For a nonconservative and nonholonomic system of Chetaev gype, the Routh equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = Q_i + \lambda_\alpha A_i^\alpha \tag{3}$$

Equation (2) is equivalent to

$$\delta L - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\delta q^i\right) + \frac{\partial L}{\partial \dot{q}^i}\left[\frac{d}{dt}(\delta q^i) - \delta \dot{q}^i\right] = 0 \tag{4}$$

Considering the relation between simultaneous variation and nonsimultaneous variation $\Delta(*) = \delta(*) + \frac{d}{dt}(*)\,\Delta t$, the last equation becomes

$$\begin{aligned}\Delta L + L\frac{d}{dt}(\Delta t) &= \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} + Q_i + \lambda_\alpha A_i^\alpha\right)\delta q^i \\ &\quad + \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\Delta q^i - \left(\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L\right)\Delta t\right] \\ &\quad + \frac{\partial L}{\partial \dot{q}^i}\left[\delta \dot{q}^i - \frac{d}{dt}(\delta q^i)\right] - Q_i\delta q^i - \lambda_\alpha A_i^\alpha \delta q^i \end{aligned} \tag{5}$$

Now we introduce Hölder's definition of variation δ_H, that is,

$$\delta_{\mathrm{H}}\dot{q}^i - \frac{d}{dt}(\delta_{\mathrm{H}} q^i) = 0, \quad A_i^\alpha \delta_{\mathrm{H}} q^i = 0 \tag{6}$$

The nonsimultaneous variation is then denoted by Δ_{H}. Among the trajectory of motion

$$\Delta_{\mathrm{H}} L + L\frac{d}{dt}(\Delta_{\mathrm{H}} t) = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}^i}\Delta_{\mathrm{H}} q^i - \left(\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L\right)\Delta_{\mathrm{H}} t\right] - Q_i\delta_{\mathrm{H}} q^i \tag{7}$$

Taking an integral of it with time t from t_1 to t_2 and noticing that

$$\left[\Delta_{\mathrm{H}} L + L\frac{d}{dt}(\Delta_{\mathrm{H}} t)\right]dt = \Delta_{\mathrm{H}}(L\,dt) \tag{8}$$

then

$$\begin{aligned}\int_{t_1}^{t_2}\Delta_{\mathrm{H}}(L\,dt) &= \left[\frac{\partial L}{\partial \dot{q}^i}\Delta_{\mathrm{H}} q^i - \left(\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L\right)\Delta_{\mathrm{H}} t\right]\Bigg|_{t_1}^{t_2} \\ &\quad - \int_{t_1}^{t_2}(Q_i\delta_{\mathrm{H}} q^i)\,dt\end{aligned} \tag{9}$$

Considering a tube on manifold $R \times TM$, satisfying the constraint conditions (1), which is constituted by the trajectaries of motion, two closed curves C_1 and

C_2 which encircle the tube are respectively represented by

$$t = t_1(\alpha), \qquad q^i = q_1^i(\alpha), \qquad \dot{q}^i = \dot{q}_1^i(\alpha) \tag{10a}$$

$$t = t_2(\alpha), \qquad q^i = q_2^i(\alpha), \qquad \dot{q}^i = \dot{q}_2^i(\alpha) \tag{10b}$$

where $\alpha(0 \le \alpha \le \rho)$ is a parameter satisfying $q^i(t,\alpha)|_{\alpha=0} = q^i(t)$, $\dot{q}^i(t,\alpha)|_{\alpha=0} = \dot{q}^i(t)$. Suppose $\alpha = 0$ and $\alpha = \rho$ represent the same point on C_1 (or C_2). Integrating Eq. (9) with respect to α over the interval $[0, \rho]$, we obtain

$$\begin{aligned}&\oint_{C_2}\left[\frac{\partial L}{\partial \dot{q}^i}\Delta_{\mathrm{H}}q^i - \left(\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L\right)\Delta_{\mathrm{H}}t\right]\\ &\quad - \oint_{C_1}\left[\frac{\partial L}{\partial \dot{q}^i}\Delta_{\mathrm{H}}q^i - \left(\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L\right)\Delta_{\mathrm{H}}t\right]\\ &= \oint_C\left(\int_{t_1}^{t_2}\Delta_{\mathrm{H}}(L\,dt)\right) + \int_{t_1}^{t_2}\left(\oint_C Q_i\delta_{\mathrm{H}}q^i\right)dt\end{aligned} \tag{11}$$

Considering relation (8), divide both sides of the last equation by $(t_2 - t_1)$ and take limit $(t_2 - t_1) \to 0$, then *an integral variant* of Poincaré-Cartan's type can be obtained

$$\begin{aligned}\frac{d}{dt}\oint_C\left[\frac{\partial L}{\partial \dot{q}^i}\Delta_{\mathrm{H}}q^i - \left(\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L\right)\Delta_{\mathrm{H}}t\right] &= \oint_C\left(\Delta_{\mathrm{H}}L + L\frac{d}{dt}(\Delta_{\mathrm{H}}t)\right)\\ &\quad + \oint_C Q_i\delta_{\mathrm{H}}q^i\end{aligned} \tag{12}$$

where $I = \oint_C\left[\frac{\partial L}{\partial \dot{q}^i}\Delta_{\mathrm{H}}q^i - \left(\dot{q}^i\frac{\partial L}{\partial \dot{q}^i} - L\right)\Delta_{\mathrm{H}}t\right]$ is called a Poincaré-Cartan's integral of a nonconservative nonholonomic system. We have proved the following:

Proposition 1. *Denote by C on manifold R × TM a closed curve encircled a tube of trajectaries of motion for a nonconservative nonholonomic constrained system. Along this curve there exists an integral variant of Poincaré-Cartan's type (Eq. 12).*

For nonholonomic constrained systems, D'Alembert-Lagrange principle is not equivalent to Hamilton action principle, that is, the equations of motion can not be derived from a stationary action principle. Thus the integral $\int_{t_1}^{t^2}\Delta_{\mathrm{H}}(L\,dt)$ is not a tatal differential of α in general. The integral $\oint_C Q_i\delta_{\mathrm{H}}q^i$ vanishes if and only if Q_i are components of conservative forces. Therefore, under the assumption of physical definition of Lagrangian and variables, there does not exist any Poincaré-Cartan's integral invariant for a nonconservative nonholonomic constrained system in general unless the system is conservative and there is a stationary action for it.

In the reference (Li and Li, 1990), the authors confused the differences between the D'Alembert-Lagrange principle (2) and Hamilton action principle for nonholonomic systems and began with the latter, took the Holder's variation as a free variation and used Routh equations to described the trajectary of motion which is in fact derived from D'Alembert-Lagrange principle. Then an integral invariant of Poincaré-Cartan's type was obtained. This process means that the system is a conservative holonomic system, which obviously conflicts with the original assumption.

3. LAGRANGIAN INVERSE PROBLEM OF NONHOLONOMIC SYSTEMS

Lagrangian inverse problem is very important for searching an integral invariant of a nonholonomic constrained system because the existence of an integral invariant is closely related to the symplectic structure of a tangent bundle which can be defined by a Lagrangian of the system.

In this section, we suppose that the system be subject to g linear nonholonomic constraints:

$$\dot{q}^{\beta} = B_{\sigma}^{\beta}(t, q^s)\dot{q}^{\sigma} + B^{\beta}(t, q^s) \tag{13}$$

This construction distinguishes two lots of coordinates, $\{q^{\sigma}\}$ and $\{q^{\beta}\}$. Assume that the configuration manifold M is of a fibration structure over a manifold M_0 of dimension $n - g + 1$ with local coordinates $\{t, q^{\sigma}\}$. The constraints determine a constraint submanifold N of 1-jet manifold $J_1 M$ (Sarlet *et al.*, 1995). The equations of motion of constrained mechanical systems, called Generalized Chaplygin's equations, are given by

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}^{\sigma}}\right) - \frac{\partial L'}{\partial \dot{q}^{\sigma}} - B_{\sigma}^{\beta}\frac{\partial L'}{\partial q^{\beta}} - t_{\sigma}^{\beta} i^{*}\left(\frac{\partial L}{\partial \dot{q}^{\beta}}\right) = 0 \tag{14}$$

where function $L' = i^{*}L \in C^2(N) \quad (i : N \to J_1 M)$ and

$$\begin{aligned} t_{\sigma}^{\beta} = {} & \left[\left(\frac{\partial B_{\sigma}^{\beta}}{\partial q^{\mu}} - \frac{\partial B_{\mu}^{\beta}}{\partial q^{\sigma}}\right) + \left(B_{\mu}^{\alpha}\frac{\partial B_{\sigma}^{\beta}}{\partial q^{\alpha}} - B_{\sigma}^{\alpha}\frac{\partial B_{\mu}^{\beta}}{\partial q^{\alpha}}\right)\right]\dot{q}^{\mu} \\ & + \left(\frac{\partial B_{\sigma}^{\beta}}{\partial t} - \frac{\partial B^{\beta}}{\partial q^{\sigma}}\right) + \left(B^{\alpha}\frac{\partial B_{\sigma}^{\beta}}{\partial q^{\alpha}} - B_{\sigma}^{\alpha}\frac{\partial B^{\beta}}{\partial q^{\alpha}}\right) \end{aligned} \tag{15}$$

In general, Eq. (14) for q^{σ} cannot be recast into the form of genuine Euler-Lagrange equations with a Lagrangian not dependent on the coordinates q^{β} because of the nonintegrability of the differential constraints, otherwise it is called *Lagrangian*.

In order to get a Lagrangian for the nonholonomic system, suppose that the dynamical equations decouple with the constraints, that is,

$$\frac{\partial L'}{\partial q^\beta} = 0, \qquad \frac{\partial B^\alpha_\sigma}{\partial q^\beta} = \frac{\partial B^\alpha}{\partial q^\beta} = 0 \tag{16}$$

It is easy to verify the following:

Theorem 2. *If the dynamical equations of a nonholonomic system decouple with the constraints and* $\Psi_\mu = t^\beta_\mu i^*(\partial L/\partial \dot{q}^\beta)$ *satisfies the Helmholtz conditions,*

$$\frac{\partial \Psi_\mu}{\partial \dot{q}_\sigma} + \frac{\partial \Psi_\sigma}{\partial \dot{q}^\mu} = 0 \tag{17a}$$

$$\frac{\partial \Psi_\mu}{\partial q^\sigma} - \frac{\partial \Psi_\sigma}{\partial q^\mu} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial \Psi_\mu}{\partial \dot{q}^\sigma} - \frac{\partial \Psi_\sigma}{\partial \dot{q}^\mu}\right) \tag{17b}$$

then this system can be reduced into a Lagrangian one.

Proof: If the conditions of the theorem are satisfied, then there exist some function $\Phi \in C^\infty(N)$ such that (Mei, 1985)

$$\Psi_\mu = \frac{d}{dt}\left(\frac{\partial \Phi}{\partial \dot{q}^\mu}\right) + \frac{\partial \Phi}{\partial q^\mu} \tag{18}$$

Therefore, Eq. (14) reduce to

$$\frac{d}{dt}\left(\frac{\partial \bar{L}}{\partial \dot{q}^\sigma}\right) - \frac{\partial \bar{L}}{\partial q^\sigma} = 0 \tag{19}$$

with a new Lagrangian $\bar{L} = L' + \Phi$.

This method can be extended to some cases where the Helmholtz conditions can not be satisfied by Ψ. If there exists a function Ψ'_μ on N satisfying the dynamical equations such that the sum $\Psi + \Psi'$ satisfies the Helmholtz conditions, the system can still be recasted into a Lagrangian one as just described.

Usually a Lagrangian system admits adjoint symmetries. If a constrained system admits some special adjoint symmetries, then an alternative Lagrangian can be realized for this system. We introduce a dynamical vector field on the constraint manifold N

$$\boldsymbol{Z} = \frac{\partial}{\partial t} + \dot{q}^\sigma \ \frac{\partial}{\partial q^\sigma} + \left(B^\beta_\sigma \dot{q}^\sigma + B^\beta\right)\frac{\partial}{\partial q^\beta} + f^\mu \frac{\partial}{\partial \dot{q}^\mu} \tag{20}$$

whose integral curves represent the solutions of Eq. (14). The *adjoint symmetries* of $\boldsymbol{Z}$ are defined by invariant 1-forms $\beta \in \bigwedge(N)$, that is, $i_{\boldsymbol{Z}}\beta = 0$, $\mathcal{L}_{\boldsymbol{Z}}\beta = 0$. If $d\beta$ is of maximum rank ε and there is a function $F \in C^\infty(N)$ such that

$$\beta = dF - S(d\boldsymbol{Z}(F)) - \boldsymbol{Z}(F)\, dt \tag{21}$$

where $S = \partial/\partial\dot{q}^\sigma \times \theta^\sigma$ is a vertical endomorphism on the bundle $\eta : N \to M$, then there exists an alternative regular Lagrangian $\boldsymbol{Z}(F)$ for the system (Sarlet, 1995).

If the mixed system of differential equations [Eqs. (13) and (14)] decouple into constraint equations and Lagrange equations for q^σ, then there exists a presymplectic submanifold $R \times TQ_0$ of constraint manifold N where $Q_0 \subset M_0$ is a configuration submanifold with local coordinates q^σ, whose presymplectic structure $\Omega^* = -d\theta_{L^*}$ is determined by the dynamical function L^* where

$$\theta^* = S(dL^*) + L^*\,dt \tag{22}$$

The dynamical equations of the system can also be derived from the variation of the action $\int_a^b (L^* \cdot \bar{\gamma})\,dt$ on such a presymplectic submanifold, where $\bar{\gamma}$ is the lift of $\gamma : [a, b] \to M_0$ to $R \times TQ_0$.

A Hamiltonian $H^* = p_\sigma \dot{q}^\sigma - L^*$ $(L^* = \mathbf{Z}(F)$ or $L' + \Phi)$ can be defined by Legendre transformation $\mathbb{F}L : R \times TQ_0 \to R \times T^*Q_0$ in the standard fashion, where the momentum is $p_\sigma = \mathbb{F}L(\dot{q}^\sigma) = \frac{\partial L^*}{\partial \dot{q}^\sigma}$. There exist a natural symplectic form

$$\omega_0 = dp_\sigma \bigwedge dq^\sigma - dH^* \bigwedge dt \tag{23}$$

and the canonical equations can be formulated by

$$i_{X_{H^*}}\omega_0 = 0, \qquad i_{X_{H^*}}dt = 0 \tag{24}$$

where $X_{H^*} = \partial/\partial t + \dot{q}^\sigma \partial/\partial q^\sigma + \dot{p}_\sigma \partial/\partial p_\sigma$ is a *Hamilton vector field* on $R \times T^*Q_0$. ω_0 is an absolute invariant 2-form of X_{H^*} because $\mathcal{L}_{X_{H^*}}\omega_0 = di_{X_{H^*}}\omega_0 + i_{X_{H^*}}d\omega_0 = 0$, and $\theta_0 = p_\sigma dq^\sigma - H^*\,dt$ is the relative invariant 1-form of X_{H^*} because $d\mathcal{L}_{X_{H^*}}\theta_0 = 0$. Without proof we have the following:

Proposition 3. *Denote by F_t the flow of X_{H^*}. c_1 and c_2 are supposed to be two closed curves encircling a tube of flow F_t on the symplectic manifold $R \times T^*Q_0$. The Poincaré-Cartan integral invariant obviously exists, that is,*

$$\oint_{c_1} p_\sigma dq^\sigma - H^*\,dt = \oint_{c_2} p_\sigma dq^\sigma - H^*\,dt \tag{25}$$

4. AN ILLUSTRATIVE EXAMPLE

Consider the problem of a vertically rolling disk on a rough horizontal plane with unit mass and radius R. Appropriate generalized coordinates are the coordinates (x, y) of the centre of mass of the disk and the azimuthal angles, angle ψ determining the position of the plane of the disk and angle φ describing its internal rotation. The condition of rolling without slipping gives rise to nonholonomic constraints of the form

$$\dot{x} = (R\,\cos\,\psi)\dot{\varphi}, \qquad \dot{y} = (R\,\sin\,\psi)\dot{\varphi}$$

with the nonzero components $B^x_\varphi = R\cos\psi$, $B^x_\varphi = R\sin\psi$.

The dimension of the manifolds Q_0 and M are 2 and 4, respectively, with notational identifications: $q^\beta = (x, y)$, $q^\sigma = (\psi, \varphi)$. The components of tensor t^β_σ read

$$t^x_\varphi = -R\dot\psi\sin\psi,\ t^x_\psi = R\dot\varphi\sin\psi,\ t^y_\varphi = -R\dot\psi\cos\psi,\ t^y_\psi = -R\dot\varphi\cos\psi,$$

The Lagrangian L and its pull back to N are, respectively,

$$L = \frac{1}{2}(\dot x^2 + \dot y^2) + \frac{1}{2}I_1\dot\varphi^2 + \frac{1}{2}I_2\dot\psi^2, \qquad L' = \frac{1}{2}(R^2 + I_1)\dot\varphi^2 + \frac{1}{2}I_2\dot\psi^2$$

where I_1 and I_2 are moments of inertia. The system obviously is a Chaplygin's system. It is easy to verify that $\Psi_\varphi = 0$, $\Psi_\psi = 0$. Therefore, Eq. (19) is satisfied by L' and the second-order equations for φ and ψ simply read

$$\frac{1}{2}(R^2 + I_1)\,\ddot\varphi = 0, \qquad I_2\ddot\psi = 0$$

with solutions $\varphi = c_1 t + c_0$, $\psi = \omega t + \psi_0$ where c_1, c_0, ω, and ψ_0 are arbitary constants. Then the solution of first-order conditions is $x = \frac{Rc_1}{\omega}\sin(\omega t + \psi_0)$, $y = -\frac{Rc_1}{\omega}\cos(\omega t + \psi_0)$.

There exists a symplectic submanifold $TQ_0 \subset N$ on which the closed two-form is defined by L'. Taking Legendre transformation $\mathbb{F}L : TQ_0 \to T^*Q_0$ in the standard fashion, the momenta are given by $p_\varphi = \partial L'/\partial\dot\varphi = (R^2 + I_1)c_1$, $p_\psi = \partial L'/\partial\dot\psi = I_2\omega$. Then the Hamiltonian is $H' = p_\varphi\dot\varphi + p_\psi\dot\psi - L' = p^2_\varphi/2(R^2 + I_1) + p^2_\psi/2I_2$. There exists a natural symplectic form on T^*Q_0

$$\omega_0 = dp_\varphi \bigwedge d\varphi + dp_\psi \bigwedge d\psi$$

which is invariant along the phase flow F_t in T^*Q_0. The corresponding 1-form $\theta_0 = p_\varphi\, d\varphi + p_\psi\, d\psi$ is a relative invariant along F_t.

Denote by c a closed curve encircling a tube of phase flow F_t on the symplectic manifold T^*Q_0 described by $c_1 = \rho_0\cos\alpha$, $c_0 = \delta_2\rho_0\sin\alpha$, $\omega = \delta_3\rho_0\cos\frac{\alpha}{2}$, $\psi_0 = \delta_4\rho_0\sin\frac{\alpha}{2}$ where ρ_0, δ_2, δ_3, and δ_4 are constants ($0 \le \alpha \le 4\pi$). Then the Poincaré integral invariant exists, that is,

$$\begin{aligned}
\oint_c p_\varphi\, d\varphi + p_\psi\, d\psi &= \oint_c (R^2 + I_1)c_1(t\,dc_1 + dc_0) + \oint_c I_2\omega\,(t\,d\omega + d\psi_0)\\
&= \int_0^{4\pi} (R^2 + I_1)\rho_0\cos\alpha(-\rho_0 t\sin\alpha + \delta_2\rho_0\cos\alpha)\,d\alpha\\
&\quad + \frac{1}{2}\int_0^{4\pi} I_2\delta_3\rho_0\cos\frac{\alpha}{2}\left(-t\delta_3\rho_0\sin\frac{\alpha}{2} + \delta_4\rho_0\cos\frac{\alpha}{2}\right)d\alpha\\
&= 2\pi(R^2 + I_1)\delta_2\rho_0^2 + \pi I_2\delta_3\delta_4\rho_0^2
\end{aligned}$$

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REFERENCES

Arnold, V. I. (1978). *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York.

Cardin, F. and Favretti, M. (1996). On Nonholonomic and Vaknomic Dynamics of Mechanical Systems with Nonintegralble Constraints, *Journal of Geometry and Physics* **18**, 295–325.

Cariñena, J. F. and Rañada, M. F. (1999). Helmholtz Conditions and alternative Lagrangians: Study of an Integrable Hénon-Heiles Systems, *International Journal of Theoretical Physics* **38**, 2049–2061.

Guo, Y. X., Shang, M., and Mei, F. X. (1999). Poincaré-Cartan Integral Invariants of Nonconservative Dynamical Systems, *International Journal of Theoretical Physics* **38**, 1017–1027.

Li, Z. P. and Li, X. (1990). Generalized Noether Theorem and Poincaré Invariant for Nonconservative Nonholonomic Systems, *International Journal of Theoretical Physics* **29**, 765–771.

Liu, D., Luo, Y., and Xin, S. Y. (1991). About the Basic Integral Variants of Holonomic Nonconservative Dynamical Systems, *Acta Mechanica Sinica* **7**, 178–185.

Marle, C. M. (1998). Various Approaches to Conservative and Nonconservative Nonholonomic Systems, *Reports on Mathematical Physics* **42**, 211–229.

Marsden, J. E. (1994). *Introduction to Mechanics and Symmetry*, Springer-Verlag, New York.

Mei, F. X. (1985). *Foundations of Mechanics of Nonholonomic Systems*, Press of Beijing Institute of Technology, Beijing (in Chinese).

Mei, F. X. (2000). On the Integration Method of Nonholonomic Dynamics, *International Journal of Non-Linear Mechanics* **35**, 229–238.

Morando, P. and Vignolo, S. (1998). A Geometric Approach to Constrained Mechanical Systems, Symmetris and Inverse Problems, *Journal of Physics A: Mathematical and General* **31**, 8233–8245.

Sarlet, W., Cantrijn, F., and Saunders, D. J. (1995). A geometrical Framework for the Study of Nonholonomic Lagrangian Systems, *Journal of Physics A: Mathematical and General* **28**, 3253–3268.